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# Solving Boundary Value Problem of a Wave Equation with Variable Coefficients

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## Abstract

In this work, we solve the problem of boundary value of a wave equation with variable coefficients

$$u_{xx}(x,t)+p(x)\mathrm{u}_x(x,t)+q(x)\mathrm{u}_t(x,t)+r(x)u(\mathrm{x,t})-\frac{1}{V^2(x)}u_{tt}(x,t)=0,$$
$$0<x<\infty,\ t\geq 0$$

Where $u(\mathrm{x,t})$ is the functions that represents the wave, $p,q,r$ and $V$ are continuous functions on the interval $(0,\infty)$, and $V(x)\neq 0$ for any $x\in(0,\infty)$. That satisfies the boundary conditions $u_x(0,t)=A(t)$, $u(\infty,t)=0$, and initial conditions $u(x,0)=u_t(x,0)=0$, solving the problem using the method of separating variables was difficult. Therefore, we found another way, which is to replace the wave equation with a system of first- order partial differential equations then solve it and we show that the solution of this system equivalent to the solution of a wave equation, using this method very helpful in getting a solution for this problem.

**Keywords:** Wave equation, boundary conditions, initial conditions, system of first-order partial differential equations.

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# حل مسألة القيمة الحدية لمعادلة الموجة ذات المعاملات المتغيرة

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**الملخص**

في هذا العمل قمنا بحل مسألة القيم الحدية لمعادلة الموجة ذات المعاملات المتغيرة بالصورة:

$$u_{xx}(x,t)+p(x)u_x(x,t)+q(x)u_t(x,t)+r(x)u(x,t)-\frac{1}{V^2(x)}u_{tt}(x,t)=0,$$

$$0<x<\infty,\ t\geq 0$$

حيث $u(x,t)$ هي الدالة التي تمثل الموجة والدوال $V, r, q, p$ دوال مستمرة على الفترة $(0,\infty)$ و $V(x) \neq$ لكل $x\in(0,\infty)$ والتي تحقق الشروط الحدية $u_x(0,t)=A(t),\ u(\infty,t)=0$ والشروط الابتدائية $u(x,0)=u_t(x,0)=0$ حل هذه المسألة باستخدام طريقة فصل المتغيرات كان امرا صعبا ولذلك اوجدنا طريقة اخرى للحل وهي استبدال معادلة الموجة بمنظومة من المعادلات التفاضلية الجزئية من الرتبة الأولى ثم نقوم بحلها وبذلك يكون حل هذا النظام يعادل حل معادلة الموجة، وكان استخدام هذه الطريقة مفيد جدا في الحصول على الحل لهذه المسألة.

**الكلمات الدالة:** معادلة الموجة، شروط حدية، شروط ابتدائية، منظومة المعادلات التفاضلية الجزئية من الرتبة الأولى.

## 1. Introduction

The wave equation is a second-order partial differential equation that describes the propagation of waves in a medium. It is commonly used to model various physical phenomena, such as vibrations in a string, sound waves, and electromagnetic waves. In many practical situations, the coefficients in the wave equation may vary with respect to space or time. These variable coefficients can significantly affect the behavior of the waves and make the problem more challenging to solve. When solving the wave equation with variable coefficients, one encounters what is known as a boundary value problem [1].

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A boundary value problem involves finding a solution to a differential equation that satisfies certain conditions at the boundaries of the domain. In the case of the wave equation, these boundary conditions typically specify the behavior of the wave at the endpoints of the domain or at certain points within the domain.
Solving a boundary value problem of the wave equation with variable coefficients requires applying appropriate techniques and methods. One commonly used approach is to separate variables by assuming a solution of the form $u(x,t)=X(x)T(t)$, where $X(x)$ represents the spatial part of the solution and $T(t)$ represents the temporal part. This separation allows us to transform the partial differential equation into a set of ordinary differential equations, which can be solved separately [2].
The specific techniques used to solve the resulting partial differential equations depend on the nature of the variable coefficients and the boundary conditions. For example, if the coefficients are constant and the boundary conditions are homogeneous, one can use methods like Fourier series or Laplace transforms to find the solution for more details see references [3-7].
It is difficult to reach the solution of this equation in the general form. The method of separation of variables is very helpful for solving linear PDEs of $2^{nd}$ order. However, it has limitations. For problems with non-constant coefficients, the method will not work and for this reason we searched for another way to solve the problem of the boundary values of Eq. (1).
[8] used the Adomian decomposition method (ADM) to solve wave equations, and compared the obtained solution by (ADM) with the Reduced Differential Transform Method and the Variational Iteration Method. [9] presented the exact solution of reduced wave equation with a variable coefficient by the solution of a classic Riccati differential equation.
[10] gave an analytical solution of a fractional wave equation for a vibrating string with Caputo time fractional derivatives. They obtained the exact solution in terms of three parameter Mittag-Leffler function.
[11] proposed a new formulation of boundary-value problem for a

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one-dimensional wave equation in a rectangular domain in boundary conditions are given on the whole boundary. They proved the well posedness of boundary-value problem in the classical and generalized senses.
[12] showed how characteristic coordinates, or equivalently how the well- known formula of d′Alembert, can be used to solve initial-boundary value problems for wave equations on fixed, bounded intervals involving Robin type of boundary conditions with time-dependent coefficients

## 2. Solution method

The wave equation with variable coefficients is

$$u_{xx}(x,t)+p(x)u_x(x,t)+q(x)u_t(x,t)+r(x)u(x,t)-\frac{1}{V^2(x)}u_{tt}(x,t)=0, \quad 0<x<\infty,\ t\geq 0 \qquad (1)$$

where $u(x,t)$ is the functions that represents the wave, $p,q,r$ and $V$ are continuous functions on the interval $(0,\infty)$, and $V(x)\neq 0$ for any $x\in(0,\infty)$. That satisfies the boundary conditions

$$u_x(0,t)=A(t),\ u(\infty,t)=0 \qquad (2)$$

And initial conditions $u(x,0)=u_t(x,0)=0$ (3)

Eq. (1) can be write as the following system of first- order partial differential equations:

$$\frac{\partial u(x,t)}{\partial x}+\frac{1}{V(x)}\frac{\partial u(x,t)}{\partial t}+\alpha_1(x)u(x,t)+\beta_1(x)v(x,t)=0 \qquad (4)$$

$$\frac{\partial v(x,t)}{\partial x}-\frac{1}{V(x)}\frac{\partial v(x,t)}{\partial t}+\alpha_2(x)u(x,t)+\beta_2(x)v(x,t)=0 \qquad (5)$$

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we can also write this system in the form of Eq. (1) if the following conditions are met.

$$p(x)=\alpha_1(x),\ q(x)=\frac{d}{dx}\left(\frac{1}{V(x)}\right)-\frac{\alpha_1(x)}{V(x)},\ r(x)=\frac{d\alpha_1(x)}{dx}-\beta_1(x)\alpha_2(x)$$

and $\frac{d\beta_1(x)}{dx}-\beta_1(x)\beta_2(x)=0$ (6)

We can verify this as follows:

Let's differentiate Eq. (4) with respect to $x$ once and with respect to $t$ again

$$\frac{\partial^2 u}{\partial x^2}+\frac{1}{V(x)}\frac{\partial^2 u}{\partial t\partial x}+\frac{d}{dx}\left(\frac{1}{V(x)}\right)\frac{\partial u}{\partial t}+\alpha_1(x)\frac{\partial u}{\partial x}+\frac{d\alpha_1(x)}{dx}u(x,t)+\beta_1(x)\frac{\partial v}{\partial x}+\frac{d\beta_1(x)}{dx}v(x,t)=0 \quad (7)$$

$$\frac{\partial^2 u}{\partial x\partial t}+\frac{1}{V(x)}\frac{\partial^2 u}{\partial t^2}+\alpha_1(x)\frac{\partial u}{\partial t}+\beta_1(x)\frac{\partial v}{\partial t}=0 \quad (8)$$

Multiplying Eq. (8) by $\frac{1}{V(x)}$, we obtain

$$\frac{1}{V(x)}\frac{\partial^2 u}{\partial x\partial t}+\frac{1}{V^2(x)}\frac{\partial^2 u}{\partial t^2}+\frac{\alpha_1(x)}{V(x)}\frac{\partial u}{\partial t}+\frac{\beta_1(x)}{V(x)}\frac{\partial v}{\partial t}=0 \quad (9)$$

Subtracting Eq. (9) from Eq. (7), we get

$$\frac{\partial^2 u}{\partial x^2}+\left[\frac{d}{dx}\left(\frac{1}{V(x)}\right)-\frac{\alpha_1(x)}{V(x)}\right]\frac{\partial u}{\partial t}+\alpha_1(x)\frac{\partial u}{\partial x}-\frac{1}{V^2(x)}\frac{\partial^2 u}{\partial t^2}+\beta_1(x)\frac{\partial v}{\partial x}$$
$$-\frac{\beta_1(x)}{V(x)}\frac{\partial v}{\partial t}+\frac{d\alpha_1(x)}{dx}u(x,t)+\frac{d\beta_1(x)}{dx}v(x,t)=0 \quad (10)$$

Multiplying Eq. (5) by $\beta_1(x)$, we get

$$\beta_1(x)\frac{\partial v(x,t)}{\partial x}-\frac{\beta_1(x)}{V(x)}\frac{\partial v(x,t)}{\partial t}=-\beta_1(x)\alpha_2(x)u(x,t)$$
$$-\beta_1(x)\beta_2(x)v(x,t) \quad (11)$$

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Substituting in Eq. (11) into Eq. (10) we get

$$\frac{\partial^2 u}{\partial x^2}+\alpha_1(x)\frac{\partial u}{\partial x}+\left[\frac{d}{dx}\left(\frac{1}{V(x)}\right)-\frac{\alpha_1(x)}{V(x)}\right]\frac{\partial u}{\partial t}+\left(\frac{d\alpha_1(x)}{dx}-\beta_1(x)\alpha_2(x)\right)u(x,t)$$

$$+\left(\frac{d\beta_1(x)}{dx}-\beta_1(x)\beta_2(x)\right)v(x,t)-\frac{1}{V^2(x)}\frac{\partial^2 u}{\partial t^2}=0 \qquad (12)$$

By comparing Eq. (12) with Eq. (1), we deduce the conditions given in (6).

Now we are looking for the solution to system (4) and (5) as follows:

$$u(x,t)=\sum_n C_{1n}e^{\mu_n(x)}e^{in(t-\tau)} \qquad (13)$$

$$v(x,t)=\sum_n C_{2n}e^{\mu_n(x)}e^{in(t+\tau)} \qquad (14)$$

where $\mu_n(x)$ functions in $x$ and $\tau=\int_0^x\frac{d\xi}{V(\xi)}$ (15)

The compensation for (13) and (14) in the system (4) and (5) we got

$$\sum_n\left[C_{1n}\left(\frac{d\mu_n}{dx}+\alpha_1\right)e^{in(t-\tau)}+C_{2n}\beta_1e^{in(t+\tau)}\right]e^{\mu_n(x)}\equiv 0$$

$$\sum_n\left[C_{1n}\alpha_2e^{in(t-\tau)}+C_{2n}\left(\frac{d\mu_n}{dx}+\beta_2\right)e^{in(t+\tau)}\right]e^{\mu_n(x)}\equiv 0 \qquad (16)$$

This means that:

$$\left(\frac{d\mu_n}{dx}+\alpha_1\right)e^{in(t-\tau)}C_{1n}+\beta_1e^{in(t+\tau)}C_{2n}=0 \qquad (17)$$

$$\alpha_2e^{in(t-\tau)}C_{1n}+\left(\frac{d\mu_n}{dx}+\beta_2\right)e^{in(t+\tau)}C_{2n}=0 \qquad (18)$$

For the system (17), (18) to have a non-zero solution, it must fulfill the condition:

$$\begin{vmatrix} \frac{d\mu_n}{dx}+\alpha_1 & \beta_1 \\ \alpha_2 & \frac{d\mu_n}{dx}+\beta_2 \end{vmatrix} = 0 \tag{19}$$

Thus, we obtain:

$$\frac{d\mu_n}{dx} = \frac{1}{2}\left[-(\alpha_1+\beta_2) \pm \sqrt{(\alpha_1+\beta_2)^2 + 4(\alpha_2\beta_1 - \alpha_1\beta_2)}\right] \tag{20}$$

Here it becomes clear that there are two values of the function $\mu_n(x)$, which are not dependent on $n$. The solution of Eq. (1) can be written as the following:

$$u(x,t) = \sum_n \left[D_{1n}e^{\mu_1(x)} + D_{2n}e^{\mu_2(x)}\right]e^{in(t-\tau)} \tag{21}$$

Since the value of the function $u(x,t)$ at a point $x$ and time $t$ can be expressed by superposition the values of this function at different points $x_i$ and time $t$, multiplied in coefficients only depend on $x$, it produces that:

$$u(x,t) = a(x)u(0,t-\tau) + b(x)u(\infty,t-\tau) \tag{22}$$

Then use $u(\infty,t) = 0$, we have

$$u(x,t) = a(x)u(0,t-\tau) \tag{23}$$

Applying the boundary condition $u_x(0,t) = A(t)$, as follows

$$u_x(0,t) = \left.\frac{da(x)}{dx}\right|_{x=0} u(0,t) - \frac{a(0)}{V(0)}\frac{du(0,t)}{dt} = A(t) \tag{24}$$

We get

$$\frac{du(0,t)}{dt} - \delta u(0,t) = -\frac{V(0)}{a(0)}A(t) \tag{25}$$

Were

$$\delta = \frac{V(0)}{a(0)}\left.\frac{da(x)}{dx}\right|_{x=0} \tag{26}$$

Solution to Eq. (25) is

$$u(0,t) = -\frac{V(0)}{a(0)}e^{\delta t}\int_0^t A(\zeta)e^{-\delta\zeta}d\zeta \tag{27}$$

So, we get the function: $u(x,t)=a(x)u(0,t-\tau)$
which satisfies the boundary conditions (2) and the initial conditions (3), but does not fulfill Eq. (1), and with that the solution that fulfills Eq. (1) we can obtain it by means of the auxiliary function $w(t)$ which is a solution to Eq. (25). Thus, in order for $u(x,t)$ to satisfy Eq. (1), it must be written in the form

$$u(x,t)=\phi_0(x)a(x)w(t-\tau)+\phi_1(x)\int_0^{t-\tau}w(s_1)ds_1+\phi_2(x)\int_0^{t-\tau}\int_0^{s_2}w(s_1)ds_1ds_2$$
$$+\phi_3(x)\int_0^{t-\tau}\int_0^{s_3}\int_0^{s_2}w(s_1)ds_1ds_2ds_3+\dots$$
$$+\phi_n(x)\int_0^{t-\tau}\int_0^{s_n}\int_0^{s_{n-1}}\dots\int_0^{s_2}w(s_1)ds_1ds_2ds_3\dots ds_n+\dots \qquad (28)$$

Also (see[7])

$$\int_0^{t-\tau}\int_0^{s_n}\int_0^{s_{n-1}}\dots\int_0^{s_2}w(s_1)ds_1ds_2ds_3\dots ds_n=\int_0^{t-\tau}\frac{(t-\tau-s)^{j-1}}{(j-1)!}w(s)ds \qquad (29)$$

We can write $u(x,t)$ form:

$$u(x,t)=\phi_0(x)a(x)w(t-\tau)+\sum_{j=1}^{\infty}\phi_j(x)\int_0^{t-\tau}\frac{(t-\tau-s)^{j-1}}{(j-1)!}w(s)ds \qquad (30)$$

where the function $w(t)$ satisfies the equation:

$$\frac{dw(t)}{dt}-\delta w(t)=-\frac{V(0)}{a(0)}A(t),\quad w(0)=0 \qquad (31)$$

Now, we will show that u (x, t) is the solution of Eq. 1.
From Eq. (30) we calculate the partial derivative with respect to $x$ and we have

$$\frac{\partial u}{\partial x}=\frac{d\phi_0(x)}{dx}a(x)w(t-\tau)+\phi_0(x)\frac{da(x)}{dx}w(t-\tau)-\frac{\phi_0(x)a(x)}{V(x)}\frac{\partial w(t-\tau)}{\partial(t-\tau)}$$
$$-\frac{\phi_1(x)}{V(x)}w(t-\tau)+\sum_{j=1}^{\infty}\frac{d\phi_j(x)}{dx}\int_0^{t-\tau}\frac{(t-\tau-s)^{j-1}}{(j-1)!}w(s)ds$$
$$-\sum_{j=2}^{\infty}\frac{\phi_j(x)}{V(x)}\int_0^{t-\tau}\frac{(t-\tau-s)^{j-2}}{(j-2)!}w(s)ds \qquad (32)$$

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We calculate the second partial derivative with respect to $x$, and we get

$$\frac{\partial^2 u}{\partial x^2}=\frac{d^2}{dx^2}(\phi_0(x)a(x))w(t-\tau)-\left[\frac{2}{V(x)}\frac{d}{dx}(\phi_0(x)a(x))\right.$$
$$\left.+\frac{d}{dx}(\frac{1}{V(x)})\phi_0(x)a(x)\right]\frac{\partial w(t-\tau)}{\partial(t-\tau)}-\left[\frac{2}{V(x)}\frac{d\phi_1(x)}{dx}+\frac{d}{dx}(\frac{1}{V(x)})\phi_1(x)\right]$$
$$.w(t-\tau)+\frac{\phi_0(x)a(x)}{V^2(x)}\frac{\partial^2 w(t-\tau)}{\partial(t-\tau)^2}+\frac{\phi_1(x)}{V^2(x)}\frac{\partial w(t-\tau)}{\partial(t-\tau)}+\frac{\phi_2(x)}{V^2(x)}w(t-\tau)$$
$$+\sum_{j=1}^{\infty}\frac{d^2\phi_j}{dx^2}\int_0^{t-\tau}\frac{(t-\tau-s)^{j-1}}{(j-1)!}w(s)ds-\sum_{j=2}^{\infty}\left[\frac{2}{V(x)}\frac{d\phi_j(x)}{dx}+\frac{d}{dx}(\frac{1}{V(x)})\phi_j(x)\right]$$
$$.\int_0^{t-\tau}\frac{(t-\tau-s)^{j-2}}{(j-2)!}w(s)ds+\sum_{j=3}^{\infty}\frac{\phi_j(x)}{V^2(x)}\int_0^{t-\tau}\frac{(t-\tau-s)^{j-3}}{(j-3)!}w(s)ds \qquad (33)$$

We also calculate both the first and second partial derivatives with respect to $t$ and we have

$$\frac{\partial u}{\partial t}=\phi_0 a\frac{\partial w(t-\tau)}{\partial(t-\tau)}+\phi_1 w(t-\tau)+\sum_{j=2}^{\infty}\phi_j\int_0^{t-\tau}\frac{(t-\tau-s)^{j-2}}{(j-2)!}w(s)ds$$
$$\frac{\partial^2 u}{\partial t^2}=\phi_0 a\frac{\partial^2 w(t-\tau)}{\partial(t-\tau)^2}+\phi_1\frac{\partial w(t-\tau)}{\partial(t-\tau)}+\phi_2 w(t-\tau)+\sum_{j=3}^{\infty}\phi_j\int_0^{t-\tau}\frac{(t-\tau-s)^{j-3}}{(j-3)!}w(s)ds$$
$$(34)$$

Compensation for (32), (33) and (34) in Eq. (1) we get

$$\left[\left(\frac{d^2}{dx^2}(\phi_0 a)+p(x)\frac{d}{dx}(\phi_0 a)+r(x)\phi_0 a\right)-\left(\frac{2}{V(x)}\frac{d\phi_1}{dx}\right.\right.$$
$$\left.\left.+\frac{d}{dx}(\frac{1}{V(x)})\phi_1+\frac{p(x)}{V(x)}\phi_1-q(x)\phi_1\right)\right]w(x-\tau)-\left[\frac{2}{V(x)}\frac{d}{dx}(\phi_0 a)\right.$$
$$\left.+\frac{d}{dx}(\frac{1}{V(x)})\phi_0 a+\frac{p(x)}{V(x)}\phi_0 a-q(x)\phi_0 a\right]\frac{\partial w(x-\tau)}{\partial(x-\tau)}$$
$$+\sum_{j=2}^{\infty}\left[\left(\frac{d^2\phi_{j-1}}{dx^2}+p(x)\frac{d\phi_{j-1}}{dx}+r(\mathrm{x})\phi_{j-1}\right)-\left(\frac{2}{V(x)}\frac{d\phi_j}{dx}+\frac{d}{dx}(\frac{1}{V(x)})\phi_j\right.\right.$$

$$+\frac{p(x)}{V(x)}\phi_j - q(x)\phi_j\Bigg)\Bigg]\int_0^{t-\tau}\frac{(t-\tau-s)^{j-2}}{(j-2)!}w(s)ds = 0 \tag{35}$$

Then we put each of all the coefficients for the following:

$$\frac{\partial w(x-\tau)}{\partial(x-\tau)} \quad ,w(x-\tau) \quad , \int_0^{t-\tau}\frac{(t-\tau-s)^{j-2}}{(j-2)!}w(s)dx$$

in Eq. (35) are equal to zero, we find that

$$\left[\frac{2}{V(x)}\frac{d}{dx}+\frac{d}{dx}\left(\frac{1}{V(x)}\right)+\frac{p(x)}{V(x)}-q(x)\right]\phi_0(x)a(x)=0 \tag{36}$$

$$\left[\frac{2}{V(x)}\frac{d}{dx}+\frac{d}{dx}\left(\frac{1}{V(x)}\right)+\frac{p(x)}{V(x)}-q(x)\right]\phi_1(x)-\left[\frac{d^2}{dx^2}+p(x)\frac{d}{dx}+r(x)\right]\phi_0(x)a(x)=0 \tag{37}$$

$$\left[\frac{2}{V(x)}\frac{d}{dx}+\frac{d}{dx}\left(\frac{1}{V(x)}\right)+\frac{p(x)}{V(x)}-q(x)\right]\phi_j(x)-\left[\frac{d^2}{dx^2}+p(x)\frac{d}{dx}+r(x)\right]\phi_{j-1}(x)=0 \tag{38}$$

Let's put

$$L=\left[\frac{2}{V(x)}\frac{d}{dx}+\frac{d}{dx}\left(\frac{1}{V(x)}\right)+\frac{p(x)}{V(x)}-q(x)\right] \tag{39}$$

$$D=\left[\frac{d^2}{dx^2}+p(x)\frac{d}{dx}+r(x)\right] \tag{40}$$

Therefore, Eq. (36) is written in the form

$$L\phi_0(x)a(x)=0 \tag{41}$$

Also, Eq. (37) is in the form

$$L\phi_1(x)=D\phi_0(x)a(x) \tag{42}$$

From Eq. (38) we get

$$L\phi_j(x)=D\phi_{j-1}(x) \quad , \quad j=2,3,4,\ldots \tag{43}$$

Now we substitute in the boundary condition: $u_x(0,t)=A(t)$ In Eq. (35), we find that

$$A(t)=\phi_0(0)\left[\left.\frac{da(x)}{dx}\right|_{x=0}w(t)-\frac{a(0)}{V(0)}\frac{dw(t)}{dt}\right]+\left[\left.\frac{d\phi_0(x)}{dx}\right|_{x=0}a(0)\right.$$

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$$-\frac{\phi_1(0)}{V(0)}\Bigg]w(t)+\sum_{j=2}^{\infty}\left[\frac{d\phi_{j-1}(x)}{dx}\Bigg|_{x=0}-\frac{\phi_j(0)}{V(0)}\right]\int_0^t\frac{(t-s)^{j-2}}{(j-2)!}w(s)ds \tag{44}$$

From (24), we get

$$A(t)=\phi_0(0)\,\mathrm{A(t)}+\left[\frac{d\phi_0(x)}{dx}\Bigg|_{x=0}a(0)-\frac{\phi_1(0)}{V(0)}\right]w(t)+\sum_{j=2}^{\infty}\left[\frac{d\phi_{j-1}(x)}{dx}\Bigg|_{x=0}-\frac{\phi_j(0)}{V(0)}\right]\int_0^t\frac{(t-s)^{j-2}}{(j-2)!}w(s)ds \tag{45}$$

Equal the coefficients of both sides of Eq. (45), we get the following:

$$\phi_0(0)=1$$

$$\phi_1(0)=\mathrm{a(0)\,V(0)}\frac{d\phi_0(x)}{dx}\Bigg|_{x=0}$$

$$\phi_j(0)=\mathrm{V(0)}\frac{d\phi_{j-1}(x)}{dx}\Bigg|_{x=0},\quad j=2,3,\ldots \tag{46}$$

Thus, we have obtained equations in $\phi_j(x)$ with the following boundary conditions:

$$\begin{cases} L\phi_0(x)a(x)=0, & \phi_0(0)=1 \\ L\phi_1(x)=D\phi_0(x)a(x), & \phi_1(0)=a(0)V(0)\frac{d\phi_0(x)}{dx}\Big|_{x=0} \\ L\phi_j(x)=D\phi_{j-1}(x), & \phi_j(0)=\mathrm{V(0)}\frac{d\phi_{j-1}(x)}{dx}\Big|_{x=0},\ j=2,3,\ldots \end{cases} \tag{47}$$

where

$$L=\left[\frac{2}{V(x)}\frac{d}{dx}+\frac{d}{dx}\left(\frac{1}{V(x)}\right)+\frac{p(x)}{V(x)}-q(x)\right]$$

and $D=\left[\frac{d^2}{dx^2}+p(x)\frac{d}{dx}+r(x)\right]$

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Finally, solution to the Eq. (1) with the conditions (2) and (3) is in the form

$$u(x,t)=\phi_0(x)a(x)w(t-\tau)+\sum_{j=1}^{\infty}\phi_j(x)\int_0^{t-\tau}\frac{(t-\tau-s)^{j-1}}{(j-1)!}w(s)ds$$

where $w(t)$ satisfies the differential Eq. (31), and the coefficients $\phi_j(x),\ j=0,1,2,...$ are given by differential equations solutions (47).

**3. Conclusions:**

In this paper, the problem of the limit values of the wave equation with variable coefficients was solved under certain boundary and initial conditions using the technique of replacing replaced by it with a system of first-order partial differential equations then solve this system and showed that have same solution.

We concludes that method was very useful in obtaining a solution to this problem.

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